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ON A DIFFERENTIAL ENCOUNTER GAME*

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A game of the encounter of two objects subject to viscous friction and control forces is examined. The sufficient conditions for the equality of the game's value to the programmed maximin are obtained under constraints of a general form.

A positional encounter game is described by differential equations with constraints on the admissible controls (U_j is a compactum)

$$x^{\cdot j} = y^j, \quad y^{\cdot j} = -k_j y^j + u^j, \quad j = 1, 2 \qquad (1)$$
$$x^j, y^j, u^j \in E^n; \; u^j \in U_j, \; k_j \geqslant 0$$

by the termination time T and by a payoff functional minimizable by the first player and maximizable by the second

$$I(z^1(\cdot), z^2(\cdot)) = \| x^2(T) - x^1(T) \|, \quad z^j = (x^j, y^j) \qquad (2)$$

The formalization of the game is completed by the concepts and constructions in /1, 2/: position strategies, constructive motions, and game value.

Let $G_T = (-\infty, T] \times E^{4n}$, $\gamma_T(t_0, z_0^1, z_0^2)$ be the value of game (1), (2) from the initial position $(t_0, z_0^1, z_0^2) \in G_T$, $X_T^j(t_0, z_0^j)$ be the set of points $x^j = x^j(T)$ in E^n which all possible motions $z^j(\cdot), z^j(t_0) = z_0^j$, can hit at instant $t = T$. We introduce into consideration the quantity (the programmed maximin)

$$\varepsilon_T(t_0, z_0^1, z_0^2) = \max_{x^2 \in X_T^2(t_0, z_0^2)} \; \min_{x^1 \in X_T^1(t_0, z_0^1)} \| x^2 - x^1 \|$$

It is required to find the conditions under which

$$\gamma_T(t_0, z_0^1, z_0^2) = \varepsilon_T(t_0, z_0^1, z_0^2) \; \forall \, (t_0, z_0^1, z_0^2) \in G_T \qquad (3)$$

In the isotropic case, i.e., when

$$U_j = \{u^j \in E^n \mid \| u^j \| \leqslant F_j\}, \quad j = 1, 2 \qquad (4)$$

a complete solution of the game is given in /3/; the sufficient conditions for (3) to be satisfied in this case have been given in /1, 2, 4/. We remark that the results mentioned do not carry over directly to the case of arbitrary U_j.

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Below, to study this problem, we use the fundamental construction in /2/ — the stable bridge. We denote by $z^j(t) = z^j(t, t_*, z_*^j, u^j(\cdot))$ the j-th player's motion corresponding to the control $u^j(t), t \geqslant t_*$, and to the initial condition $z^j(t_*) = z_*^j$. Let $z_*^j = (x_*^j, y_*^j), u^j(t) = u_*^j = \text{const}$. Then

$$x^j(T, t_*, z_*^j, u_*^j) = A(k_j, T - t_*) u_*^j + a(k_j, T - t_*) y_*^j + x_*^j$$

$$A(k,t) = \frac{t}{k} + \frac{e^{-kt}}{k^2} - \frac{1}{k^2}, \quad a(k,t) = \frac{1}{k} - \frac{e^{-kt}}{k}$$

The functions $A(k, t)$ and $a(k, t)$ are non-negative, monotonically non-decreasing in t.

Lemma. For every $t > 0$, $\Delta t \in (0, t]$, the function λ: $(0, \infty) \to E^1$, $\lambda(k) = A(k, t - \Delta t)/A(k, t)$ is monotonically increasing.

Proof. We have

$$\lambda(k) = \frac{(t - \Delta t)^2 f(k)}{t^2 g(k)}, \quad f(k) = \int_0^k \int_0^\tau e^{\omega(\Delta t - t)} d\omega\, d\tau$$

$$g(k) = \int_0^k \int_0^\tau e^{-\omega t} d\omega\, d\tau$$

It can be shown that $g(k) > 0$, $g'(k) > 0$, $g''(k) > 0$ $\forall k \in (0, \infty)$. We require the identities

$$\left(\frac{f(k)}{g(k)}\right)' = \frac{g'(k)}{g(k)}\left(\frac{f'(k)}{g'(k)} - \frac{f(k)}{g(k)}\right), \qquad \left(\frac{f'(k)}{g'(k)}\right)' = \frac{g''(k)}{g'(k)}\left(\frac{f''(k)}{g''(k)} - \frac{f'(k)}{g'(k)}\right) \tag{5}$$

We have

$$\frac{f''(k)}{g''(k)} = e^{k\Delta t}, \quad \frac{f'(k)}{g'(k)} = \int_0^k e^{-t\omega} e^{\Delta t\omega} d\omega \left(\int_0^k e^{-t\omega} d\omega\right)^{-1} < e^{k\Delta t} = \frac{f''(k)}{g''(k)}$$

Hence from the second identity in (5) it follows that $(f'(k)/g'(k))' > 0$. The function $f'(k)/g'(k)$ increases monotonically in the domain $0 < k < \infty$, and hence

$$f'(\tau) < g'(\tau) \frac{f'(k)}{g'(k)} \forall \tau \in (0, k)$$

Hence

$$\frac{f(k)}{g(k)} = \int_0^k f'(\tau)\, d\tau \left(\int_0^k g'(\tau)\, d\tau\right)^{-1} < \frac{f'(k)}{g'(k)}$$

From the first identity in (5) we have $(f(k)/g(k))' > 0$ and the monotonic increase in the function $\lambda(k)$. The lemma has been proved.

We need the following properties of the motions.

1°. Let $S = \{s \in E^n: \|s\| = 1\}$ be the unit sphere, $\delta_j: S \to E^n$, $\delta_j(s) = \max(s, u^j)$ over $u^j \in U_j$ be the support function of the convex closure of the set U_j. We take $s \in S$ and we put

$$(s, x^j(T, t_*, z_*^j, u_s^j)) = \max_{u^j \in U_j} (s, x^j(T, t_*, z_*^j, u_*^j)) = \tag{6}$$
$$A(k_j, T - t_*)\delta_j(s) + a(k_j, T - t_*)(s, y_*^j) + (s, x_*^j)$$

From (6) it follows that u_s^j does not depend on t_*, z_*^j, T. As was shown in /1/

$$\varepsilon_T(t_*, z_*^1, z_*^2) = \max\{0, \varkappa_T(t_*, z_*^1, z_*^2)\}$$
$$\varkappa_T(t_*, z_*^1, z_*^2) = \max_{s \in S}(s, x^2(T, t_*, z_*^2, u_s^2) - x^1(T, t_*, z_*^1, u_s^1))$$

2°. Let

$$\Delta t \in (0, T - t_*], \ u_*^j, u^j \in U_j$$
$$u_\Delta^j(t) = \begin{cases} u_*^j, & t \in [t_*, t_* + \Delta t) \\ u^j, & t \in [t_* + \Delta t, T] \end{cases}$$

Then

$$x^j(T, t_*, z_*^j, u_\Delta^j) = \lambda_j x^j(T, t_*, z_*^j, u^j) + (1 - \lambda_j) x^j(T, t_*, z_*^j, u_*^j) \tag{7}$$

$$\lambda_j = A(k_j, T - t_* - \Delta t)/A(k_j, T - t_*) \tag{8}$$

Theorem 1. Let $k_1 \geqslant k_2$. Then (3) is satisfied for game (1), (2).

Proof. Using the barrier properties of stable bridges (/2/, p.61), it can be shown that (3) holds if and only if for every $(t_0, z_0^1, z_0^2) \in G_T$ the set

$$W_T = W_T(t_0, z_0^1, z_0^2) = \{(t, z^1, z^2) \in G_T: \varepsilon_T(t, z^1, z^2) \leqslant \varepsilon_T(t_0, z_0^1, z_0^2)\}$$

is u^1-stable. Let $(t_*, z_*^1, z_*^2) \in W_T$. The second player selects arbitrary $\Delta t \in (0, T - t_*]$, $u_*^2 \in U_2$ and informs the first player of this. The first player chooses $u_*^1 \in U_1$ from the condition

$$\max_{s\in S}(s, x^2(T, t_*, z_*^2, u_*^2) - x^1(T, t_*, z_*^1, u_*^1)) = \|x^2(T, t_*, z_*^2, u_*^2) - x^1(T, t_*, z_*^1, u_*^1)\| = \min_{u^1\in U_1}\|x^2(T, t_*, z_*^2, u_*^2) - x^1(T, t_*, z_*^1, u^1)\| \tag{9}$$

Let $z^j(t_* + \Delta t, t_*, z_*^j, u_*^j) = z_*^j + \Delta z^j, j = 1, 2, u_s^j$ be defined by condition (6) and

$$u_\Delta{}^j(t) = \begin{cases} u_*^j, & t \in [t_*, t_* + \Delta t) \\ u_s^j, & t \in [t_* + \Delta t, T] \end{cases}$$

By virtue of the lemma, $\lambda_1 \geqslant \lambda_2$, where λ_j is defined by (8). Using this inequality, as well as (6)−(9), we have

$$\begin{aligned}
&\varkappa_T(t_* + \Delta t, z_*^1 + \Delta z^1, z_*^2 + \Delta z^2) = \\
&\max_{s\in S}(s, x^2(T, t_* + \Delta t, z_*^2 + \Delta z^2, u_s^2) - x^1(T, t_* + \Delta t, z_*^1 + \Delta z^1, \\
&u_s^1)) = \max_{s\in S}(s, x^2(T, t_*, z_*^2, u_\Delta{}^2(\cdot)) - x^1(T, t_*, z_*^1, u_\Delta{}^1(\cdot))) = \\
&\max_{s\in S}(s, \lambda_2 x^2(T, t_*, z_*^2, u_s^2) + (1 - \lambda_2) x^2(T, t_*, z_*^2, u_*^2) - \\
&\lambda_1 x^1(T, t_*, z_*^1, u_s^1) - (1 - \lambda_1) x^1(T, t_*, z_*^1, u_*^1)) \leqslant \\
&\lambda_2 \max_{s\in S}(s, x^2(T, t_*, z_*^2, u_s^2) - x^1(T, t_*, z_*^1, u_s^1)) + \\
&(1 - \lambda_2)\max_{s\in S}(s, x^2(T, t_*, z_*^2, u_*^2) - x^1(T, t_*, z_*^1, u_*^1)) + \\
&\max_{s\in S}(\lambda_2 - \lambda_1)(s, x^1(T, t_*, z_*^1, u_s^1) - x^1(T, t_*, z_*^1, u_*^1)) \leqslant \\
&\lambda_2 \varkappa_T(t_*, z_*^1, z_*^2) + (1 - \lambda_2)\varepsilon_T(t_*, z_*^1, z_*^2) \leqslant \varepsilon_T(t_*, z_*^1, z_*^2)
\end{aligned} \tag{10}$$

Because (t_0, z_0^1, z_0^2), (t_*, z_*^1, z_*^2), u_*^2, Δt are arbitrary, this signifies the u^1-stability of the set $W_T = W_T(t_0, z_0^1, z_0^2)$ $\forall (t_0, z_0^1, z_0^2) \in G_T$. Hence (3) follows. The theorem is proved.

From the theorem it follows that (3) holds for game (1), (2) with $k_1 = k_2 = 0$ and arbitrary U_1, U_2. Let inequality (10) be satisfied. The strategy extremal to the u^1-stable bridge $W_T(t_0, z_0^1, z_0^2)$ (/2/, p.61) is the first player's optimal strategy in the game from the initial position $(t_0, z_0^1, z_0^2) \in G_T$; in this case the optimal programmed strategy $u_0^2(t) = u_0^2$ determined by the condition

$$\min_{x^1\in X_T^1(t_0, z_0^1)} \|x^2(T, t_0, z_0^2, u_0^2) - x^1\| = \varepsilon_T(t_0, z_0^1, z_0^2)$$

is available to the second player.

Note that if the first player's equations of motion have the form $x^{\cdot 1} = u^1$; $x^1, u^1 \in E^n$, while the second player's have the form $x^{\cdot 2} = u^2$; $x^2, u^2 \in E^n$ or (1), then (3) is satisfied for such an encounter game as well. This assertion can be proved by the scheme presented above.

Example 1. Let $k_j > 0, U_j$ be defined by condition (4), $j = 1, 2$. Then

$$\begin{aligned}
&X_T{}^j(t_0, z_0{}^j) = \{x^j \in E^n: \|x^j - s^j\| \leqslant R_j\} \\
&\varepsilon_T(t_0, z_0^1, z_0^2) = \max\{0, \|s^2 - s^1\| + R_2 - R_1\} \\
&(s^j = a(k_j, T - t_0) y_0{}^j + x_0{}^j, R_j = A(k_j, T - t_0) F_j, j = 1,2)
\end{aligned} \tag{11}$$

Using Theorem 1 and the result in /2/, we obtain that the second equation in (11) determines the game's value if at least one of the conditions $F_1 \geqslant F_2$ or $k_1 \geqslant k_2$ is satisfied.

Example 2. Let $k_1 > 0, k_2 = 0, U_1$ be defined by condition (4) and

$$U_2 = \mathrm{co}\{u^2 \in E^n\ u^2 = u_i^2, i = 1, 2, \ldots, m\}$$

Then $X_T{}^1(t_0, z_0{}^1)$ is described by the first equation in (11) and

$$X_T^2 (t_0, z_0^2) = \text{co} \{x^2 \in E^n: x^2 = x_i^2, \ i = 1, 2, \ldots, m\} \tag{12}$$
$$\varepsilon_T (t_0, z_0^1, z_0^2) = \max \{0, \max_{1 \leqslant i \leqslant m} \| x_i^2 - s^1 \| - R_1\}$$
$$(x_i^2 = A (k_2, T - t_0) u_i^2 + a (k_2, T - t_0) y_0^2 + x_0^2)'$$

In accordance with Theorem 1 the second equation in (12) determines the game value.

Let us consider a positional pursuit game for objects described by differential equations with constraints on the admissible constraints (1). In the case of constraints (4) on the players controls the problem is known as Pontriagin's check example. The pursuing player uses control u^1 and the evading player uses control u^2. The pursuit is reckoned complete when $x^1 = x^2$. We say that the pursuit game is solvable if for every initial position $(t_0, z_0^1, z_0^2) \in E^{4n+1}$ the first player can find a strategy $u_0^1 (t, z^1, z^2)$ guaranteeing him completion of the pursuit in a finite time. It is required to find the conditions that ensure that the pursuit game is solvable. These conditions are given by

Theorem 2. Let

$$k_2 - k_1 \leqslant 0, \ \max_{s \in S} \left(\frac{\delta_2 (s)}{k_2} - \frac{\delta_1 (s)}{k_1} \right) < 0$$

Then the pursuit game is solvable.

Proof. We have

$$\varkappa_{t_0} (t_0, z_0^1, z_0^2) = \| x_0^2 - x_0^1 \| \geqslant 0$$
$$\lim_{T \to \infty} \frac{\varkappa_T (t_0, z_0^1, z_0^2)}{T - t_0} = \max_{s \in S} \left(\frac{\delta_2 (s)}{k_2} - \frac{\delta_1 (s)}{k_1} \right) < 0$$

Hence it follows that the equation

$$\omega (t) = 0 \ (\omega (t) = \varepsilon_t (t_0, z_0^1, z_0^2))$$

has the root $t = \theta, \theta \geqslant t_0$. On the strength on Theorem 1 the set $W_\theta = W_\theta (t_0, z_0^1, z_0^2)$ is u^1-stable. By applying the strategy $u_0^1 (t, z^1, z^2)$ extremal to W_θ, the first player completes the pursuit no later than the instant $t = \theta$. The theorem has been proved.

Example 3. (Pontriagin's check example). Let the sets U_1, U_2 be described by conditions (4). Then $\delta_j (s) = F_j, j = 1, 2$. The pursuit problem is solvable if $1 \leqslant k_1/k_2 < F_1/F_2$. The present conditions are more stringent than the well-known solvability conditions for Pontriagin's check example.

Example 4. Let the sets U_1, U_2 contain an interior point and be similar, i.e., $U_1 = r U_2$, $r > 0$. Then $\delta_1 (s) = r\delta_2 (s) > 0$. The pursuit problem is solvable if $1 \leqslant k_1/k_2 < r$.

Example 5. Let

$$U_1 = \{u^1 \in E^n: f_1^1 | u_1^1 | + f_2^1 | u_2^1 | + \ldots + f_n^1 | u_n^1 | \leqslant 1\}$$
$$U_2 = \{u^2 \in E^n: \ | u_i^2 | \leqslant f_i^2, \ i = 1, 2, \ldots, n\}$$
$$f^j = (f_1^j, f_2^j, \ldots, f_n^j) > 0, \ j = 1, 2$$

Then

$$\delta_1 (s) = \max \{| s_i | / f_i^1, \ i = 1, 2, \ldots, n\}$$
$$\delta_2 (s) = f_1^2 | s_1 | + f_2^2 | s_2 | + \ldots + f_n^2 | s_n |$$

The pursuit problem is solvable if $1 \geqslant k_2 / k_1 > \| f^1 \| \cdot \| f^2 \|$.

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